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The pressure distribution around unsteady boundary layers

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The effect of an unsteady boundary layer on the pressure field around a bluff body has been investigated. It is found that in an unsteady flow the friction drag is always accompanied by a form drag whose magnitude is comparable with that of the former, and thus the pressure field around the unsteady boundary layer can be very different from that of an inviscid irrotational flow. The definition of the displacement thickness is modified accordingly and interpreted as a measure of the momentum of fluid trapped in the boundary layer rather than as the distance displaced laterally by the retardation of the flow in it. The result is consistent with previous specific numerical and analytical descriptions of these boundary-layer flows.

1. Introduction

In the boundary-layer theory, it is generally accepted that the pressure gradient $\partial p/\partial x$ along the streamwise direction can be computed from the approximate equation

$$-\frac{1}{\rho}\frac{\partial p}{\partial x}=\frac{\partial u_e}{\partial t}+u_e\frac{\partial u_e}{\partial x}, \tag{1}$$

where the x-component velocity u_e just outside the boundary layer is regarded as given (Batchelor 1967, p. 305; White 1991, p. 230). In other words, the pressure is thought to be determinable without considering the detail of the boundary-layer flow. Thus Blasius (1908) and many later investigators (see Schlichting 1979, §XV.b) analysed the initial flow around, say, a circular cylinder starting impulsively from rest to move in a constant velocity, assuming that the pressure in and around the boundary layer does not vary with time. The form drag is also thought to be zero just after the start of motion but to increase with time as the boundary layer thickens and the flow separates (Stuart 1963, §VII.7).

On the other hand, Collins & Dennis (1973) solved the full Navier–Stokes equations for the initial flow over an impulsively started circular cylinder using boundary-layer variables and showed that both the friction and pressure drag are infinite at the start of motion. Bar-Lev & Yang (1975) solved the same problem by the method of matched asymptotic expansions and got similar results on the form drag. Smith & Stansby (1988) simulated it by a Lagrangian vortex method and got drag coefficients which are in agreement with analytic results at small times. However, neither of these studies gave any physical reasons for the appearance of the form drag nor an explanation of the pressure field around the cylinder; they got results simply by integrating the vorticity gradient around the cylinder.

Strictly speaking, the boundary-layer equations do not state that the pressure around the boundary layer will be the same as that of the inviscid flow around the body, but only that it is approximately uniform across the boundary layer and can be replaced by that just outside the boundary layer. How greatly the flow in the boundary layer can change the pressure around it is another matter. It seems that Lighthill (1963, p. 87) was already aware of this about thirty years ago and understood how the developing boundary layer could affect the form drag.† However, he mentioned it only briefly and did not give any mathematical analysis. The purpose of this paper is, thus, to analyse the effect of the unsteady boundary layer on the pressure field around it in detail and to assess the unsteady form drag. In the next section, it is shown, using the momentum theorem, that in an unsteady flow the friction drag is always accompanied by a form drag whose magnitude is comparable with that of the former. The reasons for appearance of this unsteady form drag are considered in §3: the displacement velocity at the outer edge of the boundary layer perturbs the velocity potential and pressure in the outer irrotational flow, and this unsteady pressure perturbation generates the form drag. The definition of the displacement thickness is also modified so that it can cope with the unsteady external flow and is interpreted as a measure of the momentum of fluid trapped in the boundary layer rather than as the distance displaced laterally by the retardation of the flow in it. Finally, conclusions are given in §4.

2. Balance of momentum in an unsteady flow around a body

Suppose that a body in an unbounded incompressible viscous fluid with density ρ and kinematic viscosity ν has started to move rectilinearly some time ago and is now in motion with velocity $-U(t)\,\boldsymbol{k}$, where t is the time elaspsed since the start of motion. The body, then, will experience a drag $D\boldsymbol{k}$ and we want to know how this drag is transferred into the fluid. To this end the momentum theorem can be used. As a system of reference, we choose coordinates fixed in space so that the velocity at infinity is zero. For the control volume V we choose the region between a cylinder with the curved surface A_S parallel to $\boldsymbol{k}$ and plane faces of area A_F normal to $\boldsymbol{k}$ and the closed surface S which is coincident with the body surface at the instant (figure 1). The momentum theorem then gives, neglecting small viscous forces acting at the cylinder surface A_S,

$$D = -\frac{\partial}{\partial t}\int_V \rho w \,\mathrm{d}V + \int_S \rho w \boldsymbol{v}\cdot\boldsymbol{n}\,\mathrm{d}S + \int_{A_F} (p_1+\rho w_1^2 - p_2 - \rho w_2^2)\,\mathrm{d}A_F - \int_{A_S} \rho w \boldsymbol{v}\cdot\boldsymbol{n}\,\mathrm{d}A_S, \tag{2}$$

where w_1, p_1, and w_2, p_2 are the values of the $\boldsymbol{k}$-component of the velocity $\boldsymbol{v}$ and the pressure at the upstream and downstream faces respectively. On the surface S the unit normal $\boldsymbol{n}$ is into the control volume (outward from the body).

As the motion had started not long before, the vorticity generated at the solid surface has not been transferred far, but is clustered near the body and also the wake is short. Thus we can take the cylinder as big enough to enclose completely the region where the flow is rotational. Since there are no sources and sinks, the irrotational flow at large distances r from the body will be like that of a source doublet and falls off as r^{-3} in three dimensions and r^{-2} in two dimensions. Thus the total momentum flux through the surface and faces of the cylinder tends to zero as the areas A_S and A_F go to infinity. On the body surface the fluid velocity is equal to $-U\boldsymbol{k}$, the velocity of the body, and the second integral of (2) becomes zero. Thus (2) becomes

$$D = -\frac{\partial}{\partial t}\int_V \rho w \,\mathrm{d}V + \int_{A_F} (p_1 - p_2)\,\mathrm{d}A_F, \tag{3}$$

† The author is grateful to Dr S. Cowley of the University of Cambridge for kindly bringing Lighthill's writings to his attention.

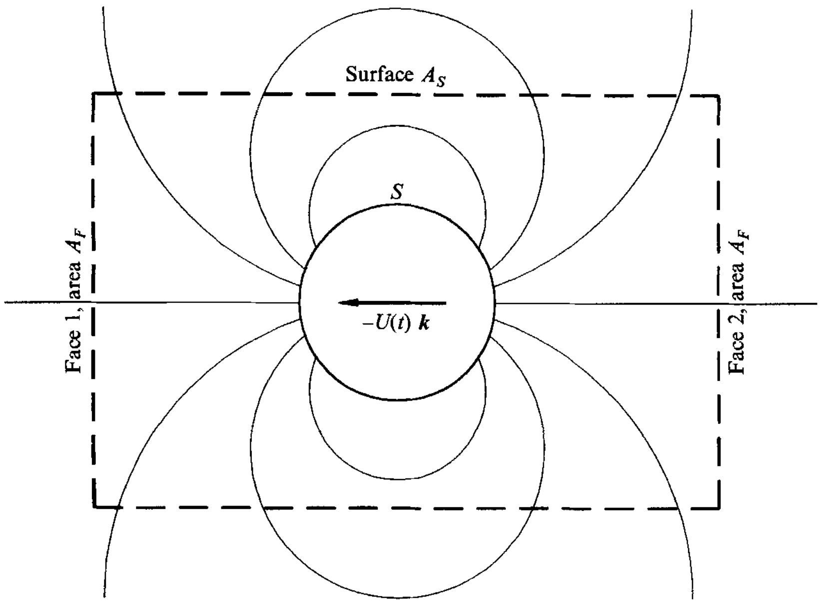

FIGURE 1. Control surface (broken line) used to obtain the drag on a body moving through a fluid.

as the control volume goes to infinity. Furthermore, if we let the length of the cylinder go to infinity first and then the area A_F of the face, the pressure integral goes to zero and we have

$$D = -\frac{\partial}{\partial t}\left(\lim_{V\to\infty}\int_V \rho w \,\mathrm{d}V\right). \tag{4}$$

Thus in an unsteady flow the drag is balanced by the increase of momentum of fluid around the body. Suppose now that the wake has not yet been formed and the displacement thickness δ_1 is relatively small. Then the total momentum of the fluid will not change if we replace the real flow around the body by uniform motion, with velocity $-U\boldsymbol{k}$, of the mass of fluid between S and the surface δ_1 distant from S and the potential flow around the virtual body formed by the body and this uniformly moving mass of fluid. Thus (4) can be written as

$$\begin{aligned} D &= \frac{\partial}{\partial t}\left(U\int_S \rho\delta_1 \,\mathrm{d}S\right) - \frac{\partial}{\partial t}\left(\lim_{V'\to\infty}\int \rho w' \,\mathrm{d}V'\right) \\ &= D_f + D_p, \end{aligned} \tag{5}$$

where the second integral is over the volume V' outside the virtual body and w' is the $\boldsymbol{k}$-component velocity of potential flow around it. The second integral in (5) gives the added mass of the virtual body multiplied by the velocity U (Milne-Thomson 1968, p. 491) and, hence, D_p denotes the increasing rate of momentum due to the growth of the added mass as well as the acceleration of the body. Considering that the pressure is approximately uniform across the boundary layer and that the reaction in the potential flow is the resultant of pressure forces around the body, we can easily anticipate that D_p is equal to the form drag and, hence, D_f is the friction drag. Thus, since the added mass is proportional to the volume of the body (in this case, of the virtual body) for the given shape (Batchelor 1967, §6.4), the form drag D_p is not zero but of the order of the friction drag D_f, even if the body moves with a constant velocity.

The fact that the actual pressure field will be different from that of the inviscid irrotational flow can be inferred again as follows. Suppose now that the body is placed in the cylindrical tube A_S with the cross-sectional area A_F and the fluid comes with uniform velocity $U\boldsymbol{k}$ from the left. Then, neglecting viscous forces acting at the cylindrical surface A_S again, we have

$$D = -\frac{\partial}{\partial t}\int_V \rho w \,\mathrm{d}V + \int_{A_F} (p_1 + \rho w_1^2 - p_2 - \rho w_2^2)\,\mathrm{d}A_F.$$

But the continuity equation gives that

$$-\frac{\partial}{\partial t}\int_V \rho w\,\mathrm{d}V = \frac{\partial(\rho U V)}{\partial t}$$

and we have, for constant U,

$$D = \int_{A_F} (p_1 + \rho w_1^2 - p_2 - \rho w_2^2)\,\mathrm{d}A_F.$$

Schwarz's inequality (Jeffreys & Jeffreys 1978, p. 54), however, gives

$$\int_{A_F} (\rho w_1^2 - \rho w_2^2)\,\mathrm{d}A_F = \int_{A_F} \rho(U^2 - w_2^2)\,\mathrm{d}A_F \leqslant 0$$

and we have

$$D \leqslant \int_{A_F} (p_1 - p_2)\,\mathrm{d}A_F,$$

or, letting the wall of the tube recede,

$$D \leqslant \lim_{A_F \to \infty} \int_{A_F} (p_1 - p_2)\,\mathrm{d}A_F. \tag{6}$$

Thus the mean pressure upstream of a body moving at a constant velocity should always be higher than that downstream of the body.

3. Drag and displacement thickness

Equation (5) shows that the drag acting on a body by the surrounding fluid and the pressure field around it are closely related to the thickening rate of the boundary layer. However, the conventional definition of the displacement thickness,

$$\delta_1(x,t) = \int_0^\infty \left(1 - \frac{u(x,y,t)}{u_e(x,t)}\right)\mathrm{d}y \tag{7}$$

(where u is the streamwise velocity parallel to the surface, u_e the external velocity, and x, y the coordinates along and normal to the surface respectively) cannot be applied when u_e varies in time. Neither can the displacement thickness be thought of as the distance through which streamlines just outside the boundary layer are displaced laterally by the retardation of the flow in the boundary layer. Thus we need a new definition and interpretation of the displacement thickness when the boundary layer is unsteady.

The new definition and corresponding interpretation of the displacement thickness can be inferred from momentum considerations again. Suppose that the cylindrical surface in figure 1 is replaced by any closed surface A with the unit outward normal $\boldsymbol{n}$. The force $\boldsymbol{F}$ acting on the body in motion with velocity $-U(t)$ by the surrounding fluid becomes

$$\boldsymbol{F} = -\frac{\partial}{\partial t}\int_V \rho \boldsymbol{v}\,\mathrm{d}V + \int_S \rho \boldsymbol{v}\boldsymbol{v}\cdot\boldsymbol{n}\,\mathrm{d}S - \int_A p\boldsymbol{n}\,\mathrm{d}A - \int_A \rho\boldsymbol{v}\boldsymbol{v}\cdot\boldsymbol{n}\,\mathrm{d}A. \tag{8}$$

Now let $\boldsymbol{v}_e$, p_e, and ϕ be the velocity, pressure, and velocity potential of the external potential flow, respectively. Then the velocity $\boldsymbol{v}$ and pressure p on the surface A, if the surface is taken sufficiently far from the body, can be replaced by the corresponding $\boldsymbol{v}_e$ and p_e and (8) becomes

$$\boldsymbol{F} = -\frac{\partial}{\partial t}\int_V \rho\boldsymbol{v}\,\mathrm{d}V - \int_A \rho\frac{\partial\phi}{\partial t}\boldsymbol{n}\,\mathrm{d}A. \tag{9}$$

In deriving (9) the relations (Batchelor 1967, p. 405),

$$\int_A \tfrac{1}{2}v_e^2\boldsymbol{n}\,\mathrm{d}A = \int_A \boldsymbol{v}_e\boldsymbol{v}_e\cdot\boldsymbol{n}\,\mathrm{d}A$$

and

$$\rho\frac{\partial\phi}{\partial t} = p_e + \frac{\rho v_e^2}{2},$$

are used. Note also that the second integral on the right-hand side of (8) is zero.

The divergence theorem gives, assuming that the potential ϕ has no singularity in the region V,

$$\int_A \rho\phi\boldsymbol{n}\,\mathrm{d}A = \int_V \rho\nabla\phi\,\mathrm{d}V + \int_S \rho\phi\boldsymbol{n}\,\mathrm{d}S$$

and (9) becomes

$$\boldsymbol{F} = -\frac{\partial}{\partial t}\left(\lim_{V\to\infty}\int_V \rho(\boldsymbol{v}-\boldsymbol{v}_e)\,\mathrm{d}V\right) - \int_S \rho\frac{\partial\phi}{\partial t}\boldsymbol{n}\,\mathrm{d}S. \tag{10}$$

Equation (4) can be rederived from (9) as follows. The integral $\int_A \rho\phi\boldsymbol{n}\,\mathrm{d}A$ is not absolutely convergent as the surface A goes to infinity, but depends on its shape. We can show, however, that it goes to zero if the surface A is a cylinder as in figure 1 and the length of the cylinder goes to infinity first. This order of integration is consistent with that to compute the drift volume and added mass of a body (Milne-Thomson 1968, §9.222). Thus we have

$$\boldsymbol{F} = -\frac{\partial}{\partial t}\left(\lim_{V\to\infty}\int_V \rho\boldsymbol{v}\,\mathrm{d}V\right),$$

or

$$\boldsymbol{F} = -\frac{\partial}{\partial t}\left(\lim_{V\to\infty}\int_V \rho(\boldsymbol{v}-\boldsymbol{v}_e)\,\mathrm{d}V\right) - \frac{\partial}{\partial t}\left(\lim_{V\to\infty}\int_V \rho\boldsymbol{v}_e\,\mathrm{d}V\right). \tag{11}$$

Equations (10) and (11) can be rewritten as

$$\boldsymbol{F} = \boldsymbol{F}_f + \boldsymbol{F}_p, \tag{12}$$

where

$$\boldsymbol{F}_f = -\frac{\partial}{\partial t}\left(\lim_{V\to\infty}\int_V \rho(\boldsymbol{v}-\boldsymbol{v}_e)\,\mathrm{d}V\right) \tag{13}$$

and

$$\boldsymbol{F}_p = -\int_S \rho\frac{\partial\phi}{\partial t}\boldsymbol{n}\,\mathrm{d}S \tag{14}$$

$$= -\frac{\partial}{\partial t}\left(\lim_{V\to\infty}\int_V \rho\boldsymbol{v}_e\,\mathrm{d}V\right). \tag{15}$$

Suppose now that the boundary layer is thin and the wake has not yet been formed. By adding the velocity $\boldsymbol{U}$ to $\boldsymbol{v}$ and $\boldsymbol{v}_e$, we can make the body stationary and the fluid arrive with velocity $\boldsymbol{U}$. Then (13) becomes

$$\boldsymbol{F}_f = -\frac{\partial}{\partial t}\left(\lim_{V\to\infty}\int_V \rho(\boldsymbol{u}-\boldsymbol{u}_e)\,\mathrm{d}V\right), \tag{16}$$

where velocities $\boldsymbol{u}$ and $\boldsymbol{u}_e$ are relative to the body, i.e.

$$\boldsymbol{u} = \boldsymbol{v}+\boldsymbol{U} \quad \text{and} \quad \boldsymbol{u}_e = \boldsymbol{v}_e + \boldsymbol{U}.$$

As the boundary layer is thin, the integral over the whole region V in (16) can be replaced by that over the volume of the shell of thickness H covering the surface S of the body, where H is small compared to the radius of curvature of the surface. The velocity $\boldsymbol{u}_e$ can be substituted by that of the inviscid flow with a vanishingly thin boundary layer, i.e. $\boldsymbol{u}_e = u_{ep}\,\boldsymbol{t}$, where u_{ep} is the velocity of this inviscid flow on the surface and $\boldsymbol{t}$ a unit vector tangential to the surface. For simplicity, let us also assume that the flow is two-dimensional or axisymmetric. Then (16) becomes, neglecting small normal velocity components of $\boldsymbol{u}$ also,

$$\boldsymbol{F}_f = \frac{\partial}{\partial t}\int_S\int_0^H \rho(u_{ep}-u)\,\boldsymbol{t}\,\mathrm{d}y\,\mathrm{d}S, \tag{17}$$

where y is the coordinate normal to the surface.

Now let the body move rectilinearly with the velocity

$$-\boldsymbol{U}(t) = -U_0 F(t)\,\boldsymbol{k},$$

where U_0 is a reference velocity and $F(t)$ a function of time. Then the external velocity $u_{ep}(x,t)$ can be written as

$$u_{ep}(x,t) = u_{e0}(x)\,F(t) = U_0 f(x)\,F(t),$$

where x is the coordinate along the surface, and the function $f(x)$ is the external velocity relative to the body when it moves with a unit velocity and depends only on the shape and orientation of the body. Thus, the *generalized displacement thickness* $\delta_1(x,t)$ can be defined as

$$\rho u_{e0}(x)\,\delta_1(x,t) = \int_0^H \rho\{u_{ep}(x,t)-u(x,y,t)\}\,\mathrm{d}y$$

$$= \rho u_{e0}(x)\int_0^H \left\{F(t)-\frac{u(x,y,t)}{u_{e0}(x)}\right\}\mathrm{d}y. \tag{18}$$

The generalized displacement thickness reduces to the conventional one when the body moves in a constant velocity, but can take both positive and negative values for oscillating flows. Equation (17) now becomes

$$\boldsymbol{F}_f = \frac{\partial}{\partial t}\int_S \rho u_{e0}(x)\,\delta_1(x,t)\,\boldsymbol{t}\,\mathrm{d}S$$

$$= \rho U_0\int_S f(x)\frac{\partial\delta_1(x,t)}{\partial t}\,\boldsymbol{t}\,\mathrm{d}S. \tag{19}$$

In other words $\rho u_{e0}\,\delta_1$ gives the momentum of fluid trapped in the boundary layer and the generalized displacement thickness δ_1 can be thought of as a measure of the momentum of this trapped fluid.

One effect of the boundary layer is to induce the transpiration (or displacement) velocity (Lighthill 1958)

$$v_f = \frac{1}{s^k}\frac{\partial}{\partial x}\left(\int_0^H s^k\{u_{ep}(x,t)-u(x,y,t)\}\,\mathrm{d}y\right)$$
$$= \frac{U_0}{s^k}\frac{\partial}{\partial x}\{s^k f(x)\,\delta_1(x,t)\}$$

just outside the boundary layer, where s is a distance from the $\boldsymbol{k}$-axis to the surface and k is zero for a two-dimensional field and one for an axisymmetric one. Owing to this displacement velocity, the external potential flow will also change and we can put

$$\phi = \phi_p + \phi_f, \tag{20}$$

where ϕ_p is the velocity potential due to the motion of the body and ϕ_f that forced by the viscous displacement velocity out of the boundary layer. The velocity potential ϕ_p can be computed as usual, while ϕ_f can be obtained by solving the Laplace equation

$$\nabla^2\phi_f = 0$$

with boundary conditions

$$-\boldsymbol{\nabla}\phi_f = 0 \quad \text{as} \quad r\rightarrow\infty,$$

and

$$-\frac{\partial\phi_f}{\partial y} = v_f \quad \text{on} \quad S,$$

where r is the distance from some material point of the body. Equation (14) thus becomes

$$\boldsymbol{F}_p = -\int_S \rho\frac{\partial\phi_p}{\partial t}\boldsymbol{n}\,\mathrm{d}S - \int_S \rho\frac{\partial\phi_f}{\partial t}\boldsymbol{n}\,\mathrm{d}S. \tag{21}$$

Though ϕ_f is $O(\delta_1)$, and thus small, $\partial\phi_f/\partial t$ which is $O(\partial\delta_1/\partial t)$ can be large: for a flow around a body set impulsively to move, say, it is $O(\nu^{\frac{1}{2}}t^{-\frac{1}{2}})$ and thus infinite just after the start of motion. Applying the unsteady Bernoulli equation but neglecting ϕ_f except for its time-derivative $\partial\phi_f/\partial t$, we also have for the pressure p around the body

$$\frac{p-p_\infty}{\rho} = \frac{\partial\phi_f}{\partial t}+\frac{\partial\phi_p}{\partial t}-\tfrac{1}{2}(\boldsymbol{\nabla}\phi_p)^2, \tag{22}$$

where p_∞ is the pressure at infinity.

When the displacement thickness δ_1 is uniform along the surface, i.e. a function of time only, expressions for $\boldsymbol{F}_f$ and $\boldsymbol{F}_p$ are simplified to

$$\boldsymbol{F}_f = \rho U_0 S_0 \frac{\mathrm{d}\delta_1(t)}{\mathrm{d}t}\boldsymbol{I}_1 \tag{23}$$

and

$$\boldsymbol{F}_p = \rho U_0\left(V_0\frac{\mathrm{d}F(t)}{\mathrm{d}t}\boldsymbol{\alpha}\cdot\boldsymbol{k}+S_0\frac{\mathrm{d}\delta_1(t)}{\mathrm{d}t}\boldsymbol{I}_2\right), \tag{24}$$

where V_0, S_0, and $\boldsymbol{\alpha}$ are the volume, surface area, and coefficient of virtual inertia of the body, respectively. $\boldsymbol{I}_1$ and $\boldsymbol{I}_2$ also depend only on the shape, orientation, and direction of motion of the body, and are defined as

$$\boldsymbol{I}_1 = \frac{1}{S_0}\int_S f(x)\,\boldsymbol{t}\,\mathrm{d}S \tag{25}$$

and
$$I_2 = -\frac{1}{S_0}\int_S \phi_A \boldsymbol{n}\,\mathrm{d}S, \tag{26}$$

where the potential ϕ_A satisfies the boundary condition

$$-\nabla\phi_A = 0 \quad \text{as} \quad r \to \infty,$$

and
$$-\frac{\partial\phi_A}{\partial y} = \frac{1}{s^k}\frac{\partial}{\partial x}\{s^k f(x)\} \quad \text{on} \quad S.$$

For a circular cylinder of radius a, these become

$$\phi_p = -U_0 a^2 \frac{\cos\theta}{r}, \quad \phi_A = -2a\frac{\cos\theta}{r}, \quad \phi_f = -2U_0 a\delta_1 \frac{\cos\theta}{r},$$

and
$$\boldsymbol{I}_1 = \boldsymbol{I}_2 = \boldsymbol{k},$$

where the azimuthal angle θ is measured from the rear stagnation point. For a sphere of radius a, we also have

$$\phi_p = -\tfrac{1}{2}U_0 a^3 \frac{\cos\theta}{r^2}, \quad \phi_A = -\tfrac{3}{2}a^2\frac{\cos\theta}{r^2}, \quad \phi_f = -\tfrac{3}{2}U_0 a^2\delta_1 \frac{\cos\theta}{r^2},$$

and
$$\boldsymbol{I}_1 = 2\boldsymbol{I}_2 = \boldsymbol{k},$$

where the colatitude θ is again measured from the rear stagnation point.

As an example, consider the flow around a body which is set impulsively to move with constant velocity $-U_0\boldsymbol{k}$. Then, for small t, the velocity relative to the wall in the boundary layer can be approximated by

$$u = u_{e0}\,\mathrm{erf}\frac{y}{2(\nu t)^{\frac{1}{2}}} \tag{27}$$

and the displacement thickness becomes

$$\delta_1 = \frac{2(\nu t)^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}. \tag{28}$$

Thus, for the pressure p_s on the surface of the cylinder, we have from (22), again neglecting $U_0\boldsymbol{k}\cdot\nabla\phi_f$ which is of the order of δ_1,

$$c_p \equiv \frac{p_s - p_\infty}{\frac{1}{2}\rho U_0^2} = 1 - 4\sin^2\theta - \frac{4}{(\pi Re\,\tau)^{\frac{1}{2}}}\cos\theta, \tag{29}$$

where
$$Re = U_0 a/\nu \quad \text{and} \quad \tau = U_0 t/a,$$

and the form drag D_p becomes

$$C_{D_p} \equiv \frac{D_p}{\rho U_0^2 a} = \frac{2\pi^{\frac{1}{2}}}{(Re\,\tau)^{\frac{1}{2}}} \tag{30}$$

which is equal to the friction drag. Similarly, we have for the pressure on the surface of the sphere

$$c_p \equiv \frac{p_s - p_\infty}{\frac{1}{2}\rho U_0^2} = 1 - \tfrac{9}{4}\sin^2\theta - \frac{3}{(\pi Re\,\tau)^{\frac{1}{2}}}\cos\theta \tag{31}$$

and for the form drag

$$C_{D_p} \equiv \frac{D_p}{\frac{1}{2}\pi\rho U_0^2 a^2} = \frac{4}{(\pi Re\, \tau)^{\frac{1}{2}}}, \tag{32}$$

which is the half of the friction drag. The last terms in (29) and (31) show that a very strong favourable pressure gradient appears along the surface of the body just after the start of motion and, due to this favourable pressure gradient, the form drag becomes infinite. Equation (30) is consistent with the leading term found by Collins & Dennis (1973) and Bar-Lev & Yang (1975).

The reacting forces on a rapidly oscillating body can also be calculated similarly. Let the velocity of the body vary as $-U_0 e^{i\omega t}\boldsymbol{k}$ where $i = \sqrt{-1}$. If both

$$St = \frac{\omega L}{U_0} \gg 1$$

and

$$St\, Re = \frac{\omega L}{U_0}\frac{U_0 L}{\nu} \gg 1,$$

where L is the characteristic length of the body, the boundary layer is thin and the tangential component of velocity in the boundary layer relative to the body surface can be approximated (Batchelor 1967, p. 354) by

$$u(y, t) = u_{e0}\, e^{i\omega t}\{1 - e^{-(1+i)\, y/\delta}\}, \tag{33}$$

where $\delta = (2\nu/\omega)^{\frac{1}{2}}$. The displacement thickness thus becomes

$$\delta_1 = \frac{\delta}{1+i} e^{i\omega t} \tag{34}$$

and from (23) and (24) we have for the force coefficient C_F of an oscillating cylinder:

$$C_F \equiv \frac{F}{\rho U_0^2 a} = i\pi\, St\, e^{i\omega t} + 2\sqrt{2}(1+i)\,\pi \left(\frac{St}{Re}\right)^{\frac{1}{2}} e^{i\omega t} \tag{35}$$

and of a sphere:

$$C_F \equiv \frac{F}{\frac{1}{2}\pi\rho U_0^2 a^2} = \tfrac{4}{3} i\, St\, e^{i\omega t} + 6\sqrt{2}(1+i)\left(\frac{St}{Re}\right)^{\frac{1}{2}} e^{i\omega t}, \tag{36}$$

where the Strouhal number St and Reynolds number Re are based on the amplitude velocity and radius, i.e.

$$St = a\omega/U_0 \quad \text{and} \quad Re = U_0 a/\nu.$$

Equations (35) and (36) are correct to the order of the boundary-layer thickness (Stuart 1966, §VII.12). Equation (35) is also consistent with the first two terms of Wang (1968), who solved the Navier–Stokes equations for the flow around an oscillating cylinder through the method of inner and outer expansions.

4. Conclusions

In this study the effect of an unsteady boundary layer on the pressure field around it has been investigated. We have shown that the momentum theorem allows us to point out some general properties concerning the pressure around the unsteady boundary layer.

(i) In an unsteady flow the friction drag is always accompanied by a form drag whose magnitude is comparable with that of the former. Thus the form drag is also infinite just after the impulsive start of motion and then decreases in inverse proportion to $\tau^{\frac{1}{2}}$.

(ii) This form drag, and the friction drag, are due to growth of the boundary layer. Increase of mass or momentum of fluid which is trapped in the boundary layer and moves together with the body appears as a friction drag, while the accompanying growth of added mass or momentum of the external potential flow appears as a form drag.

(iii) The pressure field around the unsteady boundary layer can be very different from that of inviscid irrotational flow: there appears a strong favourable pressure gradient along the surface of the body just after the impulsive start.

It is also found that the conventional definition of the displacement thickness is not adequate for an unsteady boundary layer whose external velocity is varying rapidly. The momentum consideration again helps to modify the definition of the displacement thickness and to interpret it as the measure of the momentum of fluid trapped in the boundary layer rather than as the distance displaced laterally by the retardation of the flow in it. This new concept for the displacement thickness is found useful to analyse the unsteady forces acting on bodies in motion from the surrounding fluid.

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